SOME USES OF APL IN THE THEORY OF TOURNAMENTS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Some Uses of APL in the Theory of Tournaments" submitted by Hervé-Georges Morin in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

In recent years there has been increased interest in graph theory and, in particular, in the theory of tournaments. This thesis is concerned with two models of tournaments, a "deterministic" model studied by Hartigan (2) and a "random" model studied by Narayana (7). Some relationships between these models have been clarified and numerical applications made with the help of a computer. The last part of this thesis illustrates the use of Iverson's (4) language APL for computations and simulations of tournaments with the ultimate aim of comparing knock-out and round-robin tournaments.

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INTRODUCTION

In the method of paired comparisons the basic experimental unit is the comparison of two objects, A and B, by a single judge who must choose one of them. If more than one judge is available it is easy to arrange that every judge performs every possible paired comparison. David (1) calls this situation a "balanced paired comparison experiment", corresponding to a round-robin tournament in the language of sport. Further, David mentions that the analogy between balanced paired comparison experiments and round-robin tournaments suggests that other types of tournaments might also deserve closer scrutiny. A feature of such tournaments is that balance is relaxed either for speed (i.e. fewer comparisons are made than in a round-robin tournament), or for an increased number of encounters between the most successful players, or both. Lack of balance increases the difficulty of studying the properties of these tournaments, but they have considerable intuitive appeal as a method of experimentation when our main purpose is the selection of the best object.

The random knock-out tournament, which is considered in the first chapter, is best known. The first section of chapter two deals with Hartigan's model (2) where a knock-out tournament of $n = 2^t$ players, no two of whom have the same strength, is considered. The next section describes a test of confidence for the randomness of simulations made on

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a computer. In the last chapter an indication is made of how the programming language APL (A Programming Language), developed by Iverson (4), permits a convenient and systematic study of tournaments. Then several preliminary numerical results are presented to show how the computer may be used in simulating repeated random knock-out tournaments.

CHAPTER I

RANDOM KNOCK-OUT TOURNAMENTS

1.1. Introduction

In this chapter the combinatorics of the random knock-out tournaments defined by Narayana (7) are studied further and the model where one player A, say, has probability $p \in (\frac{1}{2}, 1)$ of defeating an opponent is considered, the remaining players B_i $(i=1,2,\ldots,n-1)$ defeating each other with probability $\frac{1}{2}$. This model thus assumes that a superior player is present. Section 1.2 studies the tournament in which the number of players present is a power of two and where no byes are allowed. In Section 1.3 two other random knock-out tournaments are described, and combinatorial results similar to those of Section 1.2 are given. Following the theory developed by Narayana and Zidek (8) and the computations by Hopkins, Morin and Narayana (3), several results are introduced which apply to any random knock-out tournament as defined in (7) (we shall reproduce the definition of these tournaments in Section 1.4). The pay-offs in tournaments are studied in Section 1.5.

1.2. The Classical Case

In the classical case, $n = 2^t$ players are paired off randomly

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such that the winner may be declared after t rounds have been played. This tournament has been considered many times in the literature (cf. David (1, p. 82))and is often used in practice in such knock-out tournaments as a tennis tournament.

In the first round the $n = 2^t$ players are matched off randomly. The 2^{t-1} pairs yield 2^{t-1} losers who are eliminated from the tournament. We are left with a tournament of $2^t - 2^{t-1} = 2^{t-1}$ players. The tournament is played until only 1 player, who is the winner, is left. We shall see in Section 1.4, that any knock-out tournament can be represented by a vector; the vector for the classical case is

$$(2^{t-1}, 2^{t-2}, \ldots, 1),$$

where the elements represent the number of pairs in each round.

THEOREM 1.2.1. If R_n^i denotes the probability that the strongest player, A, plays exactly i rounds in the tournament, then

$$\left.\begin{aligned} R_n^i &= p^{i-1}q \text{ for } i < t, \\ R_n^t &= p^{t-1}, \text{ and} \\ R_n^i &= 0 \text{ for } i > t. \end{aligned}\right\} \qquad (1.1)$$

PROOF. (i) If $i < t$, the probability that the first defeat occurs at round number i equals the probability that exactly $i - 1$ suc-

cesses precede the defeat. The corresponding probability is $p^{i-1}q$.

(ii) The second equation follows since, having won $t - 1$ rounds, player A is sure of playing the last round.

(iii) If $i > t$, then $R_n^i = 0$ as there are only t rounds.

If $B_1,\ldots,B_i$ are i specified players different from A, let P_n^i be the probability that A meets $B_1,B_2,\ldots,B_i$ and perhaps other players during the tournament.

THEOREM 1.2.2. If $i \leq t$, then

$$P_n^i = \sum_{\nu=0}^{t-i-1} p^{i+\nu-1}q \frac{\binom{2^t-1-i}{\nu}}{\binom{2^t-1}{i+\nu}} + p^{t-1} \frac{\binom{2^t-1-i}{t-i}}{\binom{2^t-1}{t}} \tag{1.2}$$

PROOF. We have seen that the probability that A plays exactly $i + \nu$ rounds is $p^{i+\nu-1}q$; the probability that A plays against $B_1,\ldots,B_i$ and ν other unspecified players is

$$\frac{\binom{2^t-1-i}{\nu}}{\binom{2^t-1}{i+\nu}} .$$

A similar argument enables us to derive the last part of equation (1.2). The theorem now follows from the fact that ν can be any integer between 0 and $t - i - 1$.

Let k denote the number of rounds in the tournament, so that in the classical case, $k = t$. It will be seen in Section 1.4 that if $i \leq k$, then P_n^i is also given by the formula

$$P_n^i = \frac{1}{\binom{n-1}{i}} \sum_{\nu=1}^{k-i+1} R_n^{i+\nu-1} \binom{i+\nu-1}{i} . \tag{1.3}$$

We now derive an alternate expression for P_n^i in the classical case. First consider the case $i = 1$. It follows from (1.2) that

$$\begin{aligned} P_n^1 &= \frac{1}{n-1} (q + 2pq + 3p^2 q + \ldots + (t-1)p^{t-2} q + tp^{t-1}) \\ &= \frac{q}{n-1} (1 + 2p + \ldots + tp^{t-1}) + \frac{tp^t}{n-1} . \end{aligned} \tag{1.4}$$

If we let $X_1 = 1 + 2p + \ldots + tp^{t-1}$ and differentiate both sides of the identity

$$p + p^2 + \ldots + p^t = \frac{p(1-p^t)}{1-p}$$

with respect to p, we find that

$$X_1 = \frac{1-p^t}{(1-p)^2} - \frac{tp^t}{1-p} ;$$

therefore

$$P_n^1 = \frac{1-p^t}{(n-1)q} \quad . \tag{1.5}$$

If $p = \frac{1}{2}$, then $P_n^1 = \frac{2}{n}$. This is a general property of all random tournaments. In a knock-out tournament on n players, (n-1) losers are necessary to locate the winner. If all players are of equal strength, each of the $\binom{n}{2}$ possible pairs of players is equally likely to play in any given match. Thus the probability P_n^1 that any particular pair meets in some match of the tournament is

$$P_n^1 = \frac{(n-1)}{\binom{n}{2}} = \frac{2}{n} \quad .$$

Formula (1.3) enables us to compute P_n^i for many tournaments, but we can go further in the classical case and derive the explicit formula for P_n^i.

COROLLARY 1.2.3. If P_n^i denotes the probability that the strongest player meets i specified players in the classical case, then

$$P_n^i = \left(\frac{p}{q}\right)^{i-1} \frac{p^t}{q} \left[\frac{1}{p^t} - \binom{t}{0} - \frac{q}{p}\binom{t}{1} - \cdots - \left(\frac{q}{p}\right)^{i-1} \binom{t}{i-1}\right] \cdot \frac{1}{\binom{n-1}{i}} \quad . \tag{1.6}$$

PROOF. It follows from (1.4) that

$$\frac{n-1}{q} P_n^1 = X_1 + \frac{tp^t}{q} \quad . \tag{1.7}$$

If we apply formula (1.2), we find that

$$\frac{\binom{n-1}{2}}{pq} P_n^2 = X_2 + \frac{\binom{t}{2} p^{t-1}}{q} , \tag{1.8}$$

where

$$X_2 = 1 + 3p + \ldots + \binom{t}{2} p^{t-2} . \tag{1.9}$$

Since

$$X_1 = 1 + 2p + \ldots + tp^{t-1} ,$$

it follows that

$$X_2 - X_1 = pX_2 - \binom{t}{2} p^{t-1} - tp^{t-1} ;$$

consequently,

$$\frac{\binom{n-1}{2}}{pq} P_n^2 - \frac{\binom{n-1}{1}}{q} P_n^1 = pX_2 - \binom{t}{2} p^{t-1} - tp^{t-1} + \binom{t}{2} \frac{p^{t-1}}{q} - \frac{tp^t}{q}$$

$$= X_2 - X_1 + \binom{t}{2} \frac{p^{t-1}}{q} - \frac{tp^t}{q} .$$

It follows from (1.5) and (1.7) that

$$X_1 = \frac{1-p^t}{q^2} - \frac{tp^t}{q} ,$$

hence,

$$qX_2 = \frac{1-p^t}{q^2} - \frac{tp^t}{q} - \binom{t}{2}p^{t-1} - tp^{t-1}$$

$$= \frac{1-p^t}{q^2} - \frac{\binom{t}{1} p^t}{q}\left(1 + \frac{q}{p}\right) - \binom{t}{2} p^{t-1} ,$$

$$X_2 = \frac{1-p^t}{q^3} - \binom{t}{1} \frac{p^t}{q^2} \left[\frac{1}{p}\right] - \binom{t}{2} \frac{p^{t-1}}{q} .$$

It follows from (1.8) that

$$\frac{\binom{n-1}{2} P_n^2}{pq} = X_2 + \binom{t}{2} \frac{p^{t-1}}{q} = \frac{1-p^t}{q^3} - \binom{t}{1} \frac{p^t}{pq^2} ,$$

so

$$P_n^2 = \left(\frac{p}{q}\right) \frac{p^t}{q} \left[\frac{1}{p^t} - \binom{t}{0} - \binom{t}{1} \frac{q}{p}\right] \frac{1}{\binom{n-1}{2}} . \tag{1.10}$$

The formula

$$P_n^1 = \frac{p^t}{q} \left[\frac{1}{p^t} - \binom{t}{0}\right] \frac{1}{\binom{n-1}{1}} \tag{1.11}$$

is equivalent to Equation (1.5) and is similar to (1.10).

The special form of (1.10) and (1.11) suggests the formula

$$P_n^i = \left(\frac{p}{q}\right)^{i-1} \frac{p^t}{q}\left[\frac{1}{p^t} - \binom{t}{0} - \frac{q}{p}\binom{t}{1} - \dots - \left(\frac{q}{p}\right)^{i-1}\binom{t}{i-1}\right] \frac{1}{\binom{n-1}{i}}, \qquad (1.12)$$

where $i \leq t$. This formula certainly holds when $i = 1,2$. We now establish it in the general case by induction on i. It follows from (1.2) that

$$P_n^i = \frac{1}{\binom{n-1}{i}}\left\{p^{i-1}q + (1+i)p^i q + \dots + \binom{t-1}{i} p^{t-2} q\right\} + \frac{\binom{t}{i} p^{t-1}}{\binom{n-1}{i}}$$

and

$$P_n^{i+1} = \frac{1}{\binom{n-1}{i+1}}\left[p^i q + (i+2)p^{i+1} q + \dots + \binom{t-1}{i+1} p^{t-2} q\right] + \frac{\binom{t}{i+1} p^{t-1}}{\binom{n-1}{i+1}};$$

hence,

$$\frac{\binom{n-1}{i}}{p^{i-1}q} P_n^i = X_i + \frac{\binom{t}{i} p^{t-1}}{p^{i-1}q} \qquad (1.13)$$

and

$$\frac{\binom{n-1}{i+1}}{p^i q} P_n^{i+1} = X_{i+1} + \frac{\binom{t}{i+1} p^{t-1}}{p^i q}, \qquad (1.14)$$

where X_i and X_{i+1} are given by formulae analogous to (1.9). In fact $X_{i+1} - X_i = p\, X_{i+1} - \binom{t}{i+1} p^{t-i-1}$ since $\binom{x-1}{r-1} + \binom{x-1}{r} = \binom{x}{r}$. Therefore,

$$qX_{i+1} = X_i - \binom{t}{i+1} p^{t-i-1}. \tag{1.15}$$

Equation (1.13) and the induction hypothesis imply that

$$X_i = \frac{p^t}{q^2} \frac{1}{q^{i-1}} \left[\frac{1}{p^t} - \binom{t}{0} - \cdots - \left(\frac{q}{p}\right)^{i-1} \binom{t}{i-1} \right] - \frac{\binom{t}{i} p^{t-1}}{p^{i-1} q} ;$$

equations (1.14) and (1.15) imply that

$$\frac{\binom{n-1}{i+1}}{p^i} P_n^{i+1} = X_i - \frac{\binom{t}{i+1} p^{t-1}}{p^i} + \frac{\binom{t}{i+1} p^{t-1}}{p^i} = X_i ,$$

hence,

$$P_n^{i+1} = \left(\frac{p}{q}\right)^i \frac{p^t}{q} \left[\frac{1}{p^t} - \binom{t}{0} - \cdots - \left(\frac{q}{p}\right)^i \binom{t}{i} \right] \frac{1}{\binom{n-1}{i+1}}$$

and the proof of (1.12) is completed.

We remark that if $p = \frac{1}{2}$ then, the formula for P_n^i reduces to

$$P_n^i = \frac{2}{n \binom{n-1}{i}} \left[n - \binom{t}{0} - \binom{t}{1} - \cdots - \binom{t}{i-1} \right] .$$

The numbers $M_t^i = 2^t - \binom{t}{0} - \binom{t}{1} - \ldots - \binom{t}{i-1}$ have some interesting properties, one of which we mention here. If $i = t$ then $M_t^i = 1$, and if $i > t$ then $M_t^i = 0$. Since $\binom{t}{\nu} = \binom{t-1}{\nu} + \binom{t-1}{\nu-1}$ we have

$$M_t^i = 2^{t-1} + 2^{t-1} - \binom{t-1}{0} - \binom{t-1}{1} - \binom{t-1}{0} - \ldots - \binom{t-1}{i-1} - \binom{t-1}{i-2} ;$$

therefore,

$$M_t^i = M_{t-1}^{i-1} + M_{t-1}^i . \tag{1.16}$$

The first few values of the numbers M_t^i are given in Table 1.2.1.

$i \backslash t$	1	2	3	4	5
1	1	3	7	15	31
2	0	1	4	11	26
3	0	0	1	5	16
4	0	0	0	1	6
5	0	0	0	0	1

Table 1.2.1

1.3. Some Other Tournaments

In the last section we introduced the probabilities R_n^i and

P_n^i for the classical case where the number n of players is a power of 2. In the following section we shall study two particular tournaments called T_2 and T_4. David (1, p. 85) and others have studied the tournament T_2. The tournament T_4 is a different type of tournament and does not seem to have been studied until recently.

1.3.1. A Generalization of the Classical Case

Consider a random knock-out tournament with n players where

$$n = 2^t + k \quad (0 \leq k < 2^t) .$$

Let T_2 denote a tournament in which k preliminary matches are first played to reduce the number of players to 2^t; there are no byes in the remaining rounds as in the classical case; the winner emerges after

$$k + (2^t - 1) = n - 1 \text{ matches.}$$

THEOREM 1.3.1. If R_n^i denotes the probability that the strongest player A plays exactly i rounds in the tournament, then

$$\left.\begin{aligned} R_n^i &= p^{i-1}q \quad i < t, \\ R_n^t &= p^{t-1}q + \frac{2^t - k}{n} p^t, \text{ and} \\ R_n^{t+1} &= \frac{2k}{n} p^t. \end{aligned}\right\} \qquad (1.17)$$

PROOF. The probability that the strongest player A plays the first round and wins is $\frac{2k}{n}p$; the probability that A does not play the first round is $\frac{2^t-k}{n}$. It follows, therefore, that

$$R_n^i = \frac{2k}{n} pR_{2^t}^{i-1} + \frac{2^t-k}{n} R_{2^t}^i .$$

The values of $R_{2^t}^{i-1}$ and $R_{2^t}^i$ are given by Theorem 1.2.1. If we substitute them into the formula given above we have

$$R_n^i = \frac{2k}{n} pp^{i-2}q + \frac{2^t-k}{n} p^{i-1}q = p^{i-1}q ,$$

$$R_n^t = \frac{2k}{n} p^{t-1}q + \frac{2^t-k}{n} p^{t-1} = p^{t-1}q + \frac{2^t-k}{n} p^t ,$$

and

$$R_n^{t+1} = \frac{2k}{n} p^t ,$$

as required.

THEOREM 1.3.2. If P_n^i denotes the probability that A meets $B_1,\ldots,B_i$ and perhaps some other players, then

$$P_n^i = \left(\frac{p}{q}\right)^{i-1} \frac{p}{q}^{t}\left[\frac{1}{p^t} - \binom{t}{0} - \frac{q}{p}\binom{t}{1} \cdots - \left(\frac{q}{p}\right)^{i-1}\binom{t}{i-1}\left[1-\frac{2kq}{n}\right]\right]\frac{1}{\binom{n-1}{i}} .$$

PROOF. It follows from (1.3) that

$$P_n^i = \frac{1}{\binom{n-1}{i}} \sum_{\nu=1}^{k-i+1} R_n^{i+\nu-1} \binom{i+\nu-1}{i} ,$$

where k is the number of rounds in the tournament; in the present case $k = t + 1$. If we now apply Theorem 1.3.1 we find that

$$\binom{n-1}{i} P_n^i = \sum_{\nu=0}^{t-i-1} p^{i+\nu-1} q \binom{i+\nu}{i} + \binom{t}{i} p^{t-1} + \frac{2k}{n} \binom{t}{i-1} p^t .$$

We have already seen that

$$\binom{2^t-1}{i} P_{2^t}^i = \sum_{\nu=0}^{t-i-1} p^{i+\nu-1} q \binom{i+\nu}{i} + \binom{t}{i} p^{t-1} ,$$

so

$$\binom{n-1}{i} P_n^i = \binom{2^t-1}{i} P_{2^t}^i + \frac{2k}{n} \binom{t}{i-1} p^t .$$

Relation (1.12) states that

$$\binom{2^t-1}{i} P_{2^t}^i = (\frac{p}{q})^{i-1} \frac{p^t}{q} \left[\frac{1}{p^t} - \binom{t}{0} - \frac{q}{p} \binom{t}{1} \cdots - (\frac{q}{p})^{i-1} \binom{t}{i-1} \right] ;$$

hence,

$$\binom{n-1}{i} P_n^i = (\frac{p}{q})^{i-1} \frac{p^t}{q} \left[\frac{1}{p^t} - \binom{t}{0} - \frac{q}{p} \binom{t}{1} \cdots - (\frac{q}{p})^{i-1} \binom{t}{i-1} \left[1 - \frac{2kq}{n} \right] \right] , \quad (1.18)$$

as required.

When $p = q = \frac{1}{2}$ the formula for P_n^i reduces to

$$\binom{n-1}{i} P_n^i = \frac{2}{2^t}\left[2^t - \binom{t}{0} - \binom{t}{1} - \dots - \binom{t}{i-1}\left(1 - \frac{k}{n}\right)\right]. \qquad (1.19)$$

1.3.2. Another Type of Tournament

Let T_4 denote the tournament in which only one pair plays in each round; the winner of any round automatically plays in the next round.

THEOREM 1.3.3. If R_n^i denotes the probability that player A plays exactly i rounds then

$$\left.\begin{aligned} R_n^i &= p^{i-1}q\,\frac{n-i}{n} + \frac{p^{i-1}}{n} \quad \text{for } i < n-1 \\ R_n^{n-1} &= p^{n-2} \cdot \frac{2}{n} \end{aligned}\right\} \qquad (1.20)$$

PROOF. (i) Consider the following three mutually exhaustive and exclusive cases:

(a) A plays for the first time before round $n - i$ with the probability $\frac{n-i}{n}$;

(b) A plays for the first time in round $n - i$ with probability $\frac{1}{n}$;

(c) A plays for the first time after round $n - i$ with probability $\frac{i-1}{n}$.

The probabilities of A playing exactly i rounds are $p^{i-1}q$, p^{i-1}, 0 in cases (a), (b) (c) respectively. It follows therefore that

$$R_n^i = p^{i-1}q\,\frac{n-i}{n} + \frac{p^{i-1}}{n} \quad \text{for } i < n - 1.$$

(ii) The probability that A plays in round 1 is $\frac{2}{n}$, so

$$R_n^{n-1} = \frac{2}{n}\,p^{n-2} \quad \text{for } i = n - 1 .$$

1.4 The Theory of Random Knock-out Tournaments

Definition 1.4.1

Given an integer $n \geq 2$, we define, following (7), a random knock-out tournament with n players as a vector $(m_1,\ldots,m_k)$ of positive integers such that

$$m_1 + \ldots + m_k = n - 1, \; m_k = 1, \tag{1.21}$$

$$2m_1 \leq n$$

$$2m_i \leq n - m_1 - \ldots - m_{i-1} . \tag{1.22}$$

A tournament defined by the vector $(m_1,\ldots,m_k)$ is played as follows.

On the first round, $2m_1$ players, chosen at random from n, are paired off randomly. The remaining $n - 2m_1$ players have a "bye" for this round. The m_1 pairs yield m_1 losers who are eliminated from the tournament. We are then left with a tournament of $(n-m_1)$ players, with the vector $(m_2,\ldots,m_k)$. This inductive rule is well defined for $n > 2$, since in the case $n = 2$ there is a unique tournament of one round. The basic model chosen in this study is the one where one player A, say, has probability $p \in (\frac{1}{2},1)$ of defeating an opponent and the remaining players B_i $(i=1,\ldots,n-1)$ defeat each other with probability $\frac{1}{2}$.

In order to reproduce a theorem first proved by Narayana and Zidek (8), concerning the probability that a strong player A wins the tournament with vector $(m_1,\ldots,m_k)$, we define the vectors

$$\left.\begin{aligned} &\underline{n} = (n_1,n_2,\ldots,n_k), \text{ where} \\ &n_1 = n \text{ and} \\ &n_i = n_{i-1} - m_{i-1} \; (i \geq 2), \end{aligned}\right\} \tag{1.23}$$

and

$$\left.\begin{aligned} &\underline{p} = (p_1,p_2,\ldots,p_k), \text{ where} \\ &p_i = \frac{2m_i}{n_i}, \text{ and } q_i = 1 - p_i \quad (1 \leq i \leq k). \end{aligned}\right\} \tag{1.24}$$

Clearly n_i is the number of undefeated competitors after round i - 1, so that p_i is the probability that a specified player, among these n_i, plays round i. We also note that $n_k = 2$, from (1.21), and hence $p_k=1$.

THEOREM 1.4.1. If $\Pi = \Pi_n$ denotes the probability that A wins the tournament with vector $(m_1,\ldots,m_k)$, then

$$\Pi = (p_1p+q_1)(p_2p+q_2)\ \ldots\ (p_{k-1}p+q_{k-1})p. \tag{1.25}$$

PROOF. We will prove this theorem by induction, it can be verified directly for small values of n (say up to 4). Let us assume it is true when there are at most n - 1 players. A tournament of n players is defined by its vector $\underline{m} = (m_1,\ldots,m_k)$. The player A can survive the first round in one of the following ways:

(a) he plays the first round and wins with probability $p \times \dfrac{2m_1}{n}$; or

(b) he has a bye with probability $\dfrac{n-2m_1}{n}$.

It follows, therefore, that player A survives the first round with probability

$$p_1p + q_1 \ . \tag{1.26}$$

We are now in the case of a tournament of $n - m_1 = n_1$ players with vector $(m_2, \ldots . m_k)$. Since $n_1 \leq n - 1$ the inductive hypothesis is that

$$\Pi_{n_1} = (p_2 p + q_2) \ldots (p_{k-1} p + q_{k-1}) p ; \tag{1.27}$$

we conclude, therefore, that

$$\Pi_n = (p_1 p + q_1) \Pi_{n_1} . \tag{1.28}$$

It is clear that the coefficient of p^i in the expansion of the right hand side of (1.25) is the probability b_{k-i} that a particular player receives $k - i$ byes in winning the tournament. If we let $\underline{w} = (p, p^2, \ldots, p^k)$ and $\underline{b} = (b_{k-1}, b_{k-2}, \ldots . b_o)$, then Π can be written as an inner product of two vectors, namely,

$$\Pi = \underline{b} \cdot \underline{w}' . \tag{1.29}$$

THEOREM 1.4.2. If Π denotes the probability that A wins the tournament, and if E(R) denotes the expected number of rounds played by A, then

$$\Pi + qE(R) = 1 . \tag{1.30}$$

PROOF. The theorem is certainly true for $n = 2$. The recurrence relation

$$E_n(R) = p_1 q + p_1 p(E_{n_1}(R)+1) + q_1 E_{n_1}(R) , \tag{1.31}$$

can be established by considering what happens to player A in round 1, as in the proof of Theorem 1.4.1. Our inductive hypothesis states that

$$\Pi_{n_1} + qE_{n_1}(R) = 1 . \tag{1.32}$$

If we solve for $E_{n_1}(R)$ in (1.31) and Π_{n_1} in (1.28) and substitute the resulting expressions in (1.32), we obtain the relation

$$\frac{\Pi_n}{p_1 p+q_1} + \frac{q(E_n(R)-p_1)}{p_1 p+q_1} = 1 . \tag{1.33}$$

Therefore

$$\Pi + qE(R) = \Pi_n + qE_n(R) = p_1 p + q_1 + qp_1 = 1,$$

as required.

Now let $P_n^{(i)}$ be the probability that A meets exactly i specified players $B_1, B_2, \ldots, B_i$ in the course of the tournament.

Clearly $P_n^{(i)} = R_n^i \binom{n-1}{i}^{-1}$, where R_n^i is the probability that A plays exactly i rounds and $\binom{n-1}{i}^{-1}$ is the probability for A to meet i specified players.

Let P_n^i be the probability that A met $B_1, B_2, \ldots$ and B_i, and perhaps some other players. It follows, from the method of inclusion and exclusion, that

$$P_n^i = \sum_{\nu=0}^{k-i} R_n^{i+\nu} \frac{\binom{n-i-1}{\nu}}{\binom{n-1}{i+\nu}} , \tag{1.34}$$

or

$$P_n^i = \frac{1}{\binom{n-1}{i}} \sum_{\nu=1}^{k-i+1} R_n^{i+\nu-1} \binom{i+\nu-1}{i} . \tag{1.35}$$

Let $Q_n(x)$ denote the probability that A meets at least one of the players $B_1, \ldots, B_x$ the function $Q_n(x)$ increases from P_n^1 to 1 as x varies from 1 to n - 1. Narayana and Zidek (8) used the method of inclusion and exclusion to show that

$$Q_n(x) = \sum_{i=1}^{n-1} (-1)^{i-1} P_n^i \binom{x}{i} .$$

Consider now the case where $p = \frac{1}{2}$ and let y be the integer such that

$$Q_n(y) \geq \frac{1}{2} \text{ and } Q_n(y-1) < \frac{1}{2} ;$$

calculations on the computer suggest the conjecture that $y = \{\frac{n-1}{3}\}$

for every random knock-out tournament of the type defined in this section, where $\{x\}$ denotes the least integer not less than x. We also conjecture that $y = \{\frac{n}{3}\}$ for the tournament T_4. The number $\{\frac{n-1}{3}\}$, which is called the "stoodge", is encountered in other parts of the theory of tournaments. We hope that in the classical case the numbers M_t^i , defined in section 1.2, will be of help in proving this conjecture.

General programs illustrating the numerical calculation of R_n^i and P_n^i have been prepared in the language APL [3]; in particular, programs are available for the following tournaments, when $n = 2^t + k$ with $0 \le k < 2^t$:

T_1, the tournament with vector $\underline{m} = (m_i)$ where

$$m_i = [\frac{n + 2^{i-1} - 1}{2^i}] ,$$

T_2, the tournament with vector $(k, 2^{t-1}, \ldots, 1)$;

T_3, the tournament with vector $(1, 1, \ldots, 1)$; and

T_4, the tournament with vector $(1, 1, \ldots, 1)$ defined in 1.3, in which the winner of each round plays in the next round.

1.5 Pay-offs in Tournaments

If each player is given 1 unit for each round he enters, then

$$P_1 = E(R) = \sum_i i\, R_n^i \tag{1.36}$$

is A's expected winnings. We shall examine a few other ways of assigning rewards and investigate their relationship to P_1.

If each player is given 1 unit each time he wins a round, then A's expected pay-off is

$$P_2 = pP_1 \ , \tag{1.37}$$

since Pr (A wins exactly i rounds) = Pr (A plays exactly i rounds) × Pr (A wins the i^{th} round).

Suppose each player enters the tournament with one unit to his credit and forfeits his unit to the player who beats him, if he is beaten. We can express the expected gain P_3 of player A in terms of P_1 . If we consider separately the cases where A does or does not win the tournament, we find that

$$P_3 = \Pi P_2 + (1-\Pi)(P_2-1) \tag{1.38}$$

hence $P_3 = P_2 - (1-\Pi) = P_2 - qP_1 = (p-q)P_1$, by (1.30) and (1.37).

Suppose we give a player i units when he plays his i^{th} game.

If P_4 denotes the expected gain of A in this case, then

$$P_4 = \sum_i \frac{i(i+1)}{2} R_n^i = \sum_i \frac{i^2}{2} R_n^i + \sum_i \frac{i}{2} R_n^i = \frac{G''(1)}{2} + \frac{P_1}{2} + \frac{P_1}{2}$$

or

$$P_4 = P_1 + \frac{G''(1)}{2} \tag{1.39}$$

where $G(s)$ is the generating function of the sequence R_n^i.

Let us suppose that n players participate in a random knock-out tournament. Every player who plays in a round receives 1 unit and the winner of the tournament receives a bonus of B units. If this procedure is followed then player A can expect to win an amount equal to

$$E(R) + \Pi B .$$

If $B = q^{-1}$, then player A can expect to win an amount equal to q^{-1} no matter which random tournament is played; this follows from the fact that

$$E(R) + \Pi q^{-1} = q^{-1}(qE(R) + \Pi) = q^{-1} ,$$

by (1.30). We have not been able to decide whether there are

other constant values of B for which A's expected gain is independent of the tournament played.

CHAPTER II

DETERMINISTIC MODEL AND TEST OF RANDOMNESS

2.1. Introduction

Chapter I was devoted to a random knock-out tournament in which one strong player (when $p > \frac{1}{2}$) competed against $n - 1$ weaker players of equal strength. In the second section of this chapter we shall discuss a "deterministic" tournament in which no two players have the same strength. In the last section we use a test based on confidence intervals to check the validity of tournament simulations made with the help of a computer.

2.2. The Deterministic Model

Consider a knock-out tournament of $n = 2^t$ players no two of whom have the same strength. We assume the stronger player always wins in any encounter, so the only element of randomness arises in the order in which the players are matched. We shall label the players in order of increasing strength, so for example, player 1 is the weakest player and player 2^t is the strongest.

THEOREM 2.2.1. If N_{2^t} denotes the number of tournaments

with 2^t players, then

$$N_{2^t} = \frac{2^t!}{2^{2^t-1}} \quad . \tag{2.1}$$

PROOF. If we take into account the number of ways of matching off the players in the first round we find that

$$N_{2^t} = \{(2^t-1)(2^t-3)\ldots 3\}N_{2^{t-1}} \; ; \tag{2.2}$$

hence,

$$N_{2^t} = \frac{2^t!}{2^{t-1}!\left(2^{2^{t-1}}\right)} \times N_{2^{t-1}}$$

and the required result follows by induction.

THEOREM 2.2.2. If $P(r,i)$ denotes the probability that a player r survives round i, then

$$P(r,i) = \frac{\binom{r-1}{2^i-1}}{\binom{2^t-1}{2^i-1}} \quad . \tag{2.3}$$

PROOF. If player r is to survive round i, then the 2^i-1 players he encounters either directly or indirectly in the first i

rounds must all be weaker than he is. The probability that this is the case is given by the above formula.

THEOREM 2.2.3. If R_r^i denotes the probability that player r plays exactly i rounds, then

$$R_r^i = \frac{\binom{2^t-2^{i-1}}{2^t-r} - \binom{2^t-2^i}{2^t-r}}{\binom{2^t-1}{2^t-r}} \tag{2.4}$$

PROOF. It is not difficult to see that

$$P(r,i-1) = P(r,i) + R_r^i \tag{2.5}$$

for $i = 1,2,\ldots,2^t-1$, if we adopt the convention that $P(t,0) = 1$. The theorem now follows from (2.5) if we use the identity

$$\frac{\binom{a-b}{c}}{\binom{a}{c}} = \frac{\binom{a-c}{b}}{\binom{a}{b}} .$$

We say a player y is of type $(\alpha,\beta,\ldots,\nu)$ if y loses in round $\alpha+1$ to someone who loses in round $\beta+1\ldots$ to someone who loses in round $\nu+1$ to the strongest player, 2^t.

THEOREM 2.2.4. If y denotes the player of type $(\alpha,\beta,\ldots,\nu)$, then the expected values of y and y(y+1) are respectively

$$\left.\begin{aligned} E(y:\alpha,\beta,\ldots\nu) &= 2^t \frac{2^\nu}{2^\nu+1} \cdots \frac{2^\alpha}{2^\alpha+1} \\ E(y(y+1):\alpha,\beta,\ldots\nu) &= 2^t(2^t+1) \frac{2^\nu}{2^\nu+2} \cdots \frac{2^\alpha}{2^\alpha+2} \end{aligned}\right\} \quad (2.6)$$

PROOF. We estimate first the strength of the player beaten by the r^{th} weakest player in round i, assuming that $r \geq 2^i$ and that player r survives round i. If we let x denote the strength of the player defeated by r in round i, then the expected value of x is given by the formula

$$E(x) = \sum_{\nu=1}^{r-1} \nu\, P^*(\nu, i-1),$$

where

$$P^*(\nu,i-1) = P(\nu,i-1) \Big/ \sum_{\nu=1}^{r-1} P(\nu,i-1)$$

denotes the conditional probability that player r beats player ν in round i. It follows from theorem 2.2.2 and a well-known identity that

$$E(x) = \sum_{\nu=1}^{r-1} \nu \binom{\nu-1}{2^{i-1}-1} \Big/ \sum_{\nu=1}^{r-1} \binom{\nu-1}{2^{i-1}-1}$$

$$= 2^{i-1} \binom{r}{2^{i-1}+1} \Big/ \binom{r-1}{2^{i-1}} = r\, 2^{i-1}\, (2^{i-1}+1)^{-1} \ . \tag{2.7}$$

Notice that the ratio of the expected strength of the player beaten by player r in any round i to the strength of player r depends only on the number of the round. The expected strength of a player of type (α) will then be $\frac{2^{\alpha}}{2^{\alpha+1}}\ 2^t$ and, in general, the expected strength of the player of type $(\alpha,\beta,\ldots,\nu)$ will be

$$2^t \cdot \frac{2^{\nu}}{2^{\nu}+1} \cdots\cdots \frac{2^{\alpha}}{2^{\alpha}+1} \ . \tag{2.8}$$

The same method can be used to find the expected value of $x(x+1)$.

This result agrees with the one of Hartigan (2). Moon (6) derived this formula by deriving a recurrence relation for $P_t(i:\alpha,\ldots,\mu,\nu)$ where P_t denotes the probability that player i is of type $(\alpha,\ldots,\mu,\nu)$. The proof we have given here is based upon a suggestion of T.V. Narayana; this method can also be used to find corresponding results for players in a tournament T_4 (see for example, Narayana (9)).

2.3 A Confidence Interval for Π

Following Chapter I, we know that player A has the probability Π of winning a tournament of the type described in Section 1.4. Hence the probability distribution of the number of times player A

wins a tournament is a binomial $b(\Pi,m)$ where m is the number of repetitions (simulations) of the tournament. To test the validity of the random numbers used in the simulations we want to obtain a confidence interval of degree $1 - \alpha$ for Π.

The following well-known lemmas show the relations between the binomial distribution, the beta distribution, and the Fisher-Snedecor distribution.

LEMMA 2.3.1. If k is the number of successes in m independent trials, each of which has probability p of success, then

$$P\{k > x\} = \frac{B_p(x+1,m-x)}{B\,(x+1,m-x)}\,, \tag{2.9}$$

where $B_p(\alpha,\beta)$ is the incomplete $B(\alpha,\beta)$ distribution

$$B_p(\alpha,\beta) = \int_0^p u^{\alpha-1}(1-u)^{\beta-1}du\ .$$

PROOF. We have to show that the right hand side of (2.9) is equal to $\sum_{k=x+1}^{m} \binom{m}{k} p^k(1-p)^{m-k}$. If we integrate by parts, we find that

$$B_p(x+1,m-x) = \left[\frac{u^{x+1}(1-u)^{m-x-1}}{x+1}\right]_0^p + \frac{m-x-1}{x+1}\int_0^p u^{x+1}(1-u)^{m-x-2}du;$$

If we repeat this process we find that

$$B_p(x+1,m-x) = \sum_{\nu=1}^{m-x} p^{x+\nu}(1-p)^{m-x-\nu} \frac{(m-x-1)(m-x-2)\ldots(m-x-\nu+1)}{(x+1)(x+2)\ldots(x+\nu)}.$$

We now observe that

$$\frac{1}{x+1} = \frac{m!}{(x+1)!(m-x-1)!} \times \frac{x!(m-x-1)!}{m!} = \binom{m}{x+1} B(x+1,m-x),$$

$$\frac{m-x-1}{(x+1)(x+2)} = \binom{m}{x+2} B(x+1,m-x) ,$$

$$\frac{(m-x-1)\ldots(m-x-j+1)}{(x+1)(x+2)\ldots(x+j)} = \binom{m}{x+j} B(x+1,m-x) ,$$

so that

$$\int_0^p u^x(1-u)^{m-x-1}du = B(x+1,m-x) \sum_{k=x+1}^{m} \binom{m}{k} p^k(1-p)^{m-k} ,$$

as required.

LEMMA 2.3.2. If k is the number of successes in m independent trials, each of which has probability p of success, then

$$P\{k > x\} = Pr\ \{F \le \frac{m-x}{x+1} \cdot \frac{p}{1-p}\} \tag{2.10}$$

where F is the Fisher distribution with $2(x+1)$ and $2(m-x)$ degrees of freedom.

PROOF. The Fisher-Snedecor distribution function is defined by the relation

$$P = \Pr\{F \leq F_p\} = \int_0^{F_p} \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} F^{\frac{\nu_1}{2}-1}}{\left(1+\frac{\nu_1}{\nu_2}F\right)^{\frac{\nu_1+\nu_2}{2}}} \frac{dF}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} . \qquad (2.11)$$

Let F be equal to $\frac{\nu_2}{\nu_1}\frac{u}{1-u}$ and $dF = \frac{\nu_2}{\nu_1}\frac{du}{(1-u)^2}$; hence

$$\int_0^{u_p} u^{\frac{\nu_1}{2}-1}(1-u)^{\frac{\nu_2}{2}-1} \frac{du}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} = P ,$$

and

$$P = \frac{B_{u_p}\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \quad \text{where } u_p = \frac{\left(\frac{\nu_1}{\nu_2}\right)F_p}{1+\frac{\nu_1}{\nu_2}F_p} .$$

One can see that P is equal to the probability that $k > x$ in a binomial distribution (Lemma 2.3.1) with

$$p = u_p \;, \quad x + 1 = \frac{\nu_1}{2} \;, \text{ and } m - x = \frac{\nu_2}{2}$$

hence, $\nu_1 = 2(x+1)$, $\nu_2 = 2(m-x)$, and $F_p = \frac{m-x}{x+1} \cdot \frac{p}{1-p}$.

Therefore,

$$\Pr\{k > x\} = \Pr\{F \leq \frac{m-x}{x+1} \cdot \frac{p}{1-p}\} \;, \tag{2.12}$$

as required.

THEOREM 2.3.3. If k_o is the number of successes in m independent trials, each of which has probability p of success, then the limits p_1 and p_2 of the confidence interval, of degree $1 - \alpha$, for p are

$$\left.\begin{aligned} P_1 &= \frac{k_o}{(m-k_o+1)F_{P_2}\{2(m-k_o+1),2k_o\}+k_o} \\ &\text{and} \\ P_2 &= \frac{(k_o+1)F_{1-P_1}\{2(k_o+1),2(m-k_o)\}}{m-k_o+(k_o+1)F_{1-P_1}\{2(k_o+1),2(m-k_o)\}} \end{aligned}\right\} \tag{2.13}$$

where $P_2 - P_1 = 1 - \alpha$ and $F_p(\alpha,\beta)$ is such that $P(F \geq F_p(\alpha,\beta)) = P$ where F has the Fisher distribution with α and β degrees of freedom.

PROOF. Given that in m simulations there were k_o successes, the limits p_1 and p_2 are such that

$$\left.\begin{array}{l} P(k \leq k_o | p = p_2) = P_1 \\ \text{and} \\ P(k \geq k_o | p = p_1) = 1 - P_2 \\ \text{where } P_2 - P_1 = 1 - \alpha. \end{array}\right\} \qquad (2.14)$$

If we use Lemma 2.3.2, these relations can be expressed as

$$\left.\begin{array}{l} F_{1-P_1}\{2k_o+2, 2(m-k_o)\} = \dfrac{m-k_o}{k_o+1}\ \dfrac{p_2}{1-p_2} \\ \text{and} \\ F_{1-P_2}\{2k_o, 2(m-k_o+1)\} = \dfrac{m-k_o+1}{k_o}\ \dfrac{p_1}{1-p_1} \end{array}\right\} \qquad (2.15)$$

If in (2.11) we replace F by $\frac{1}{G}$ we have

$$P = \int_{\infty}^{\frac{1}{F_p}} \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} G^{-\frac{\nu_1}{2}+1}}{\left(1+\frac{\nu_1}{\nu_2}\frac{1}{G}\right)^{\frac{\nu_1+\nu_2}{2}}} \cdot \frac{-dG}{G^2 B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)}$$

or

$$1 - P = \int_0^{\frac{1}{F_p}} \frac{\left(\frac{\nu_2}{\nu_1}\right)^{\frac{\nu_2}{2}} G^{\frac{\nu_2}{2} - 1}}{\left(1 + \frac{\nu_2}{\nu_1} G\right)^{\frac{\nu_1+\nu_2}{2}}} \cdot \frac{dG}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)}$$

consequently,

$$F_{1-P}(\nu_2, \nu_1) = \frac{1}{F_p(\nu_1, \nu_2)} ,$$

and (2.15) is equivalent to

$$\left.\begin{aligned} & F_{1-P_1}\{2(k_o+1), 2(m-k_o)\} = \frac{m-k_o}{(k_o+1)} \cdot \frac{p_2}{1-p_2} \\ \text{and} & \\ & F_{P_2}\{2(m-k_o+1), 2k_o\} = \frac{k_o}{m-k_o+1} \cdot \frac{1-p_1}{p_1} \end{aligned}\right\} . \qquad (2.16)$$

The theorem follows if we solve (2.16) for p_1 and p_2 .

The following example gives an application of this theorem. By using random numbers, the tournament with vector (2,1) (see Section

1.4) was simulated on a computer. Player A won 113 times out of 200 simulations. Theorem 2.3.3 gives in this case

$$p_1 = .5585$$

and

$$p_2 = .6001,$$

where $1 - \alpha = .95$.

The theory of Section 1.4 gives us the value

$$\Pi = .5625$$

for the tournament with vector (2,1). Since this value of Π falls between the two limits p_1 and p_2 we can say with 95% confidence that the randomness of the simulations was well accomplished by the computer.

COROLLARY 2.3.4. For m large, the limits Π_1 and Π_2 of the confidence interval for Π are

$$\Pi_1 = \frac{k_o}{m} - u_{1-\frac{\alpha}{2}} \sqrt{\frac{\frac{k_o}{m}\left(1-\frac{k_o}{m}\right)}{m}}$$

and

$$\Pi_2 = \frac{k_o}{m} + u_{1-\frac{\alpha}{2}} \sqrt{\frac{\frac{k_o}{m}\left(1-\frac{k_o}{m}\right)}{m}} .$$

where u is a normal variate.

PROOF. For large m, the estimator $\frac{k_o}{m}$ of Π has a normal $\left(\Pi, \frac{\Pi(1-\Pi)}{m}\right)$ distribution. The inequality

$$P\left\{-u_{1-\frac{\alpha}{2}} < \frac{\hat{\Pi} - \Pi}{\sqrt{\frac{\Pi(1-\Pi)}{m}}} < u_{1-\frac{\alpha}{2}}\right\} = 1 - \alpha$$

where $\hat{\Pi} = \frac{k_o}{m}$, gives the confidence interval

$$\frac{\hat{\Pi} + \frac{u^2}{2m} - u\sqrt{\frac{\hat{\Pi}(1-\hat{\Pi})}{m} + \frac{u^2}{4m^2}}}{1 + \frac{u^2}{m}} < \Pi < \frac{\hat{\Pi} + \frac{u^2}{2m} + u\sqrt{\frac{\hat{\Pi}(1-\hat{\Pi})}{m} + \frac{u^2}{4m^2}}}{1 + \frac{u^2}{m}}$$

for Π. We can neglect the term $\frac{u^2}{m}$ if m is large so that

$$\frac{k_o}{m} - u_{1-\frac{\alpha}{2}}\sqrt{\frac{\frac{k_o}{m}\left(1-\frac{k_o}{m}\right)}{m}} < \Pi < \frac{k_o}{m} + u_{1-\frac{\alpha}{2}}\sqrt{\frac{\frac{k_o}{m}\left(1-\frac{k_o}{m}\right)}{m}} \text{ .}$$

CHAPTER III

THE USE OF APL IN THE THEORY OF TOURNAMENTS

3.1. Introduction

Chapter III is devoted to the application of APL to the computational aspect of tournaments. APL was designed by K.E. Iverson in response to a need for a language or notation which could be used on a blackboard to describe algorithms (4). APL was of particular use in our work because of the ease of learning the language and the precision with which complex expressions could be formalized and then evaluated. In this chapter we present a few sample programs from "A first APL tournament package" [3] proposed at the University of Alberta. The twenty or so programs listed in the package, together with detailed illustrations of their use, have already permitted us to make a first attempt of simulation of tournaments using APL.

3.2. APL and Tournaments

In this section we will give some programs following the theory developed in Chapter I. Let us consider first the tournament

T_2, of $n = 2^t + k$ players, studied in detail in Section 1.3. The tournament T_2 has vector

$$(k, 2^{t-1}, 2^{t-2}, \ldots, 1) \tag{3.1}$$

If R_n^i denotes the probability that player A plays exactly i rounds then by (1.17)

$$\left.\begin{aligned} R_n^i &= p^{i-1}q \qquad i < t \\ R_n^t &= p^{t-1}q + \frac{2^t - k}{n}\, p^t \\ R_n^{t+1} &= \frac{2k}{n}\, p^t \quad . \end{aligned}\right\} \tag{3.2}$$

We can program (3.2) in APL language in the following manner under the function name TWOR :

```
    ∇ V←TWOR N;T;K;A;B;C
[1]   T←0
[2]   →(N>2*T←T+1)/2
[3]   A←Q,Q×P*⍳¯2+T←T-1
[4]   B←(Q×P*T-1)+((2*T)-K←N-2*T)÷N
[5]   C←(2×K×P*T)÷S←N
[6]   V←A,B,C
    ∇
```

<u>Program I</u>

Steps (1) and (2) allow us to compute the value of t such that $0 \leq k < 2^t$. Steps (3), (4), (5) compute R_n^i $(i < t)$, R_n^t, R_n^{t+1}, respectively.

The documentation of any of the following functions is stated as a function, which, when executed, types out the description of the function and how it is used. The name of the function with the documentation is the same as the function being described but with the addition of the suffix HOW. For example, the function TWOR is documented in the function TWORHOW. Thus for (3.2) we have the following descriptive function TWORHOW:

```
    ∇ TWORHOW
[1]   '           TWOR N '
[2]   'PROBABILITIES THAT THE STRONGEST PLAYER A PLAYS'
[3]   'EXACTLY I ROUNDS IN THE TOURNAMENT T2,'
[4]   'VECTOR (K,2*T-1,2*T-2,...,1),'
[5]   'WITH PROBABILITY P=1-Q OF WINNING AGAINST ANY'
[6]   'PLAYER B[J]   I=1,2,...,T+1      J=1,2,...,N-1'
    ∇
```

<u>Program II</u>

We want now to indicate a program for the computation of

P_n^i , the probability that A meets i specified players B_j and perhaps some other players. We apply the formula (1.3)

$$P_n^i = \frac{1}{\binom{n-1}{i}} \sum_{\nu=0}^{k=i} R_n^{i+\nu} \binom{i+\nu}{i} , \tag{3.3}$$

where k is the number of rounds.

```
     ∇  V←PNI W;U;M;I
[1]     V←¯1+I←1
[2]     U←⍳M←0
[3]     U←U,W[I+M]×(I!I+M)
[4]     →((-I-⍴W)≥M←M+1)/3
[5]     V←V,(+/U)÷(I!S-1)
[6]     →((⍴W)≥I←I+1)/2
     ∇
```

Program III

The documentation of program 3 is contained under the function name PNIHOW.

```
    ∇ PNIHOW
[1]     '       PNI W           THIS PROGRAM SHOWS THE'
[2]     'PROBABILITIES THAT THE STRONGEST PLAYER A MEETS'
[3]     'I OR MORE PLAYERS IN THE TOURNAMENT, WHOSE VECTOR'
[4]     'RNI IS GIVEN BY W.'
    ∇
```

Program IV

The programs mentioned above give an idea of the content of our APL package [3]. The following are examples of programs written to compute some commonly occurring functions.

(1) A program for computing the numbers $Q_n(x)$ of the theory (see page 22) .

(2) A program for computing the numbers R_n^i in the case of the tournaments T_1 and T_3 with vectors $m_i = [\frac{n+2^{i-1}-1}{2^i}]$ and $(1,1,\ldots,1)$, respectively (see page 23).

(3) A program for computing the numbers R_n^i in the case of the special tournament T_4 (see Section 1.3).

(4) A program for comparing T_1, T_2, T_3, T_4 for a given number of players.

(5) A program for computing different pay-offs (see Section 1.5).

(6) A program for computing the numbers R_r^i of Chapter II (see (2.7)).

Our package concludes with programs for the simulation on a computer of either repeated round-robin or repeated knock-out tournaments as defined in this thesis. Of course, extensive simulation is necessary before drawing valid conclusions from these; indeed, as emphasized in the APL package, "these simulations represent very much a pilot and preliminary effort, undertaken partly as an exercice to study the capabilities of random number generation in APL"

3.3 Comparison Between Round-Robin and T_{4K} Tournaments

The round-robin tournament is defined by David (1). We let T_{4K} denote the tournament played with the rule described in Section 1.3, but with the difference that the winner of a simulation is kept in the first round of the next simulation. The Tables mentioned by David (1, pp. 113, 115) give the smallest number of replications required to ensure with at least a predetermined probability P' the selection of the best object and the values of ν for decision rule $\mathcal{Q}$ of (1, §6.3) in order to select the best player in the subset. Our table gives the number of replications required and the

percentage of times the best player was selected for both round-robin and T_{4K} tournaments.

N \ P		P = .55 REP	S	P = .60 REP	S	P = .65 REP	S	P = .70 REP	S	P = .75 REP	S
4	R_R	71	60	18	90	8	70	5	80	3	80
	T_{4K}	142	75	36	100	16	80	10	80	6	75
5	R_R	68	80	17	65	8	90	4	70	3	78.3
	T_{4K}	170	90	43	80	20	90	10	90	8	95
6	R_R	65	90	16	80	7	55	4	80	3	95
	T_{4K}	195	85	48	10	21	80	12	70	9	90
7	R_R	61	70	15	60	7	65	4	90	3	85
	T_{4K}	214	70	53	90	25	50	14	65	11	90
8	R_R	58	90	15	95	7	65	4	80	3	78.3
	T_{4K}	232	90	60	65	68	80	16	85	12	85

Table I

The above table indicates that further simulation could provide a basis for comparison betweet round-robin and knock-out tournaments, especially when we consider variants of the one outlier model (see also [10]).

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